EXISTENCE OF SHOCK-WAVE SOLUTIONS FOR SOME NON-CONSERVATIVE SYSTEMS

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ABSTRACT. In this article, we prove the existence of shock curves for two systems in nonconservative form using the product defined by Dal Maso,Lefloch, Murat in [1]. One of the systems is strictly hyperbolic with genuinely nonlinear characteristic fields while the other one has repeated eigenvalues with an incomplete set of right eigenvectors (parabolic degeneracy). The shock curves for the two systems are compared based upon a parameter k tending to 0.

1. INTRODUCTION

This article is devoted to the study of the first order non-conservative systems

$$\begin{aligned} &u_t + uu_x - \sigma_x = 0, \\ &\sigma_t + u\sigma_x = 0 \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &u_t + uu_x - \sigma_x = 0, \\ &\sigma_t + u\sigma_x - k^2 u_x = 0. \end{aligned} \tag{1.2}$$

in the domain $\Omega = \{(x,t) : -\infty < x < \infty,\ t > 0\}$. Here k is a positive constant. In order to implement the approach using the family of paths (we henceforth refer to them as DLM paths) for studying a nonconservative system (see [1],[4]), the most important step to begin with is to prove the existence of a path using which we can address the issue of solving the Riemann problem. In this article, we address this issue in regard to the systems (1.1) and (1.2).

The system (1.2) is strictly hyperbolic with the eigenvalues given by $\lambda_1(u,\sigma) = u - k,\ \lambda_2(u,\sigma) = u + k$ with the corresponding right eigenvectors $E_1(u,\sigma) = \begin{pmatrix} 1 \\ k \end{pmatrix},\ E_2(u,\sigma) = \begin{pmatrix} 1 \\ -k \end{pmatrix}$. This system arises in elastodynamics and the Riemann problem for the system was solved using Volpert's product in [2].

The system (1.1) on the contrary fails to be hyperbolic (parabolic degeneracy) since it has repeated eigenvalues $\lambda_1(u,\sigma) = u = \lambda_2(u,\sigma)$ and an incomplete set of right eigenvectors. Such systems even when in conservative form are tough to analyse due to absence of full set of eigenvectors. In [5] a class of such systems in conservative form had been studied and it was shown that singular concentrations tend to develop in one of the dependent variables. Motivated by this, existence of singular solutions to (1.1) had been studied in [3] using the method of weak asymptotics. But a seemingly more interesting question in this context is to examine the existence of a DLM path so as to be able to find a *shock-wave solution* to (1.1) for Riemann type initial data. It has been shown in [3] that the Riemann problem for

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(1.1) cannot be solved using Volpert's product.
Another interesting question would be to explore the connection between the solutions of (1.1) and (1.2) as k tends to 0.
We recall that a DLM path ϕ is a locally Lipschitz map $\phi : [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the following properties:
1. $\phi(0; v_L, v_R) = v_L$ and $\phi(1; v_L, v_R) = v_R$ for any v_L and v_R in $\mathbb{R}^2$,
2. $\phi(t; v, v) = v$, for any v in $\mathbb{R}^2$ and $t \in [0,1]$,
3. For every bounded set Ω of $\mathbb{R}^2$, there exists $k \geq 1$ such that

$$|\phi_t(t; v_L, v_R) - \phi_t(t; w_L, w_R)| \leq k|(v_L - w_L) - (v_R - w_R)|,$$

for every v_L, v_R, w_L, w_R in Ω and for almost every $t \in [0,1]$.
The paths $\phi = (\phi_1, \phi_2)$ and $\tilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2)$ given by

$$\phi(t; v_L, v_R) : \begin{cases} \phi_1(t; u_L, u_R) = \begin{cases} u_L + 2t(u_R - u_L), \ t \in [0, \frac{1}{2}], \\ u_R, \ t \in [\frac{1}{2}, 1]. \end{cases} \\ \phi_2(t; \sigma_L, \sigma_R) = \sigma_L + t(\sigma_R - \sigma_L), \ t \in [0,1]. \end{cases} \tag{1.3}$$

and

$$\tilde{\phi}(t; v_L, v_R) : \begin{cases} \tilde{\phi}_1(t; u_L, u_R) = \begin{cases} u_L + 2t(u_R - u_L), \ t \in [0, \frac{1}{2}], \\ u_R, \ t \in [\frac{1}{2}, 1] \end{cases} \\ \tilde{\phi}_2(t; \sigma_L, \sigma_R) = \begin{cases} \sigma_L, \ t \in [0, \frac{1}{2}], \\ \sigma_L + (2t-1)(\sigma_R - \sigma_L), \ t \in [\frac{1}{2}, 1] \end{cases} \end{cases} \tag{1.4}$$

where $v_L = (u_L, \sigma_L), \ v_R = (u_R, \sigma_R) \in \mathbb{R}^2$, satisfy the above conditions and are thus examples of DLM paths.
Given a system

$$v_t + A(v)v_x = 0, \quad v(x,t) \in \mathbb{R}^2, \ x \in \mathbb{R}, \ t > 0$$

in non-conservative form with Riemann type initial data:

$$v(x,0) = v_L \text{ if } x < 0, \ v(x,0) = v_R \text{ if } x > 0,$$

and a DLM path ϕ, it was shown in [1] that a shock wave solution exists with speed s if and only if v_L, v_R and s satisfy the following Rankine-Hugoniot(R-H) condition:

$$\int_0^1 \{-sI + A(\phi(t; v_L, v_R))\}\phi_t(t; v_L, v_R)dt = 0, \tag{1.5}$$

where I denotes the identity matrix. We remark that the matrix A for the systems (1.1) and (1.2) are given by $A(u,\sigma) = \begin{pmatrix} u & -1 \\ 0 & u \end{pmatrix}$ and $A(u,\sigma) = \begin{pmatrix} u & -1 \\ -k^2 & u \end{pmatrix}$ respectively.
In Section 2, we derive a condition for a DLM path to satisfy the R-H condition for the systems (1.1) and (1.2). In Section 3, we show that the curves ϕ and $\tilde{\phi}$ mentioned above satisfy the R-H condition and find the *shock curves* for the solutions of the Riemann problem for (1.1) and (1.2).

2. Condition on a DLM path and the initial data to satisfy the Rankine-Hugoniot condition

In this section, we derive a condition on a DLM path $\phi = (\phi_1, \phi_2)$ in order that it might satisfy the R-H condition (1.5) for the systems (1.1) and (1.2) for given Riemann type initial data

$$v(x,0) = (u(x,0), \sigma(x,0)) = \begin{cases} v_L = (u_L, \sigma_L), \ x < 0, \\ v_R = (u_R, \sigma_R), \ x > 0. \end{cases} \tag{2.1}$$

Lax's admissibility condition applied to the systems (1.1) and (1.2) implies that for a shock-wave type solution we must have $u_L > u_R$.

Theorem 2.1. *Given initial states $v_L = (u_L, \sigma_L)$ and $v_R = (u_R, \sigma_R)$ and a DLM path $\phi = (\phi_1, \phi_2)$, the R-H condition (1.5) for the system (1.1) is satisfied if*

$$\int_0^1 \phi_1(t; u_L, u_R)(\phi_2)_t(t; \sigma_L, \sigma_R) dt = [\sigma] . \frac{[\frac{u^2}{2}] - [\sigma]}{[u]}, \tag{2.2}$$

where $[u] = (u_R - u_L)$ denotes the jump in u and $s = \frac{[\frac{u^2}{2}] - [\sigma]}{[u]}$ is the speed of the shock.

Proof. We recall that for the system (1.1) we have $A(u, \sigma) = \begin{pmatrix} u & -1 \\ 0 & u \end{pmatrix}$. Substituting this in (1.5), we obtain $\int_0^1 \begin{pmatrix} -s + \phi_1 & -1 \\ 0 & -s + \phi_1 \end{pmatrix} \begin{pmatrix} (\phi_1)_t \\ (\phi_2)_t \end{pmatrix} = 0$. Thus we have the relations

$$\int_0^1 -s(\phi_1)_t + \phi_1(\phi_1)_t - (\phi_2)_t \ dt = 0, \tag{2.3}$$

and

$$\int_0^1 -s(\phi_2)_t + \phi_1(\phi_2)_t \ dt = 0. \tag{2.4}$$

Now (2.3) on simplification (using the fact $\phi_1(0) = u_L, \ \phi_1(1) = u_R, \ \phi_2(0) = \sigma_L, \ \phi_2(1) = \sigma_R$) gives $s = \frac{[\frac{u^2}{2}] - [\sigma]}{[u]}$. Substituting this in (2.4) we obtain

$$\int_0^1 \phi_1(t; u_L, u_R)(\phi_2)_t(t; \sigma_L, \sigma_R) dt = [\sigma] . \frac{[\frac{u^2}{2}] - [\sigma]}{[u]}.$$

$\square$

Proceeding similarly as in the above theorem, we can prove the corresponding result for the system (1.2).

Theorem 2.2. *Given initial states $v_L = (u_L, \sigma_L)$ and $v_R = (u_R, \sigma_R)$ and a DLM path $\phi = (\phi_1, \phi_2)$, the R-H condition (1.5) for the system (1.2) is satisfied if*

$$\int_0^1 \phi_1(t; u_L, u_R)(\phi_2)_t(t; \sigma_L, \sigma_R) dt = [\sigma] . \frac{[\frac{u^2}{2}] - [\sigma]}{[u]} + k^2 [u], \tag{2.5}$$

where $[u] = (u_R - u_L)$ denotes the jump in u and $s = \frac{[\frac{u^2}{2}] - [\sigma]}{[u]}$ is the speed of the shock.

Proof. A similar calculation as in the proof of Theorem 2.1 with the corresponding matrix $A(u,\sigma) = \begin{pmatrix} u & -1 \\ -k^2 & u \end{pmatrix}$ for the system (1.2) gives the result. □

3. Existence of DLM paths satisfying the Rankine-Hugoniot condition

In this section, we prove the existence of DLM paths satisfying the conditions (2.2) and (2.5) for the systems (1.1) and (1.2) respectively. In particular, we show that given a left state $v_L = (u_L, \sigma_L)$ the paths ϕ and $\tilde{\phi}$ mentioned in the introduction give rise to shock curves passing through v_L. Thus for any state $v_R = (u_R, \sigma_R)$ (with $u_R < u_L$) lying on these curves, we can solve the Riemann problem using a shock wave with speed s as given in the previous section.

Theorem 3.1. *Given a left state $v_L = (u_L, \sigma_L)$, the DLM path ϕ defined in* (1.3) *gives a shock-wave solution for the system* (1.1) *with the right states $v_R = (u_R, \sigma_R)$ on the shock curve*

$$S_1 : \sigma = \sigma_L - \frac{1}{4}(u - u_L)^2,\ u < u_L. \tag{3.1}$$

Proof. The result follows from a straightforward calculation by substituting ϕ_1 and ϕ_2 from (1.3) in (2.2). □

Similar to the above theorem for the system (1.2) we have the following result.

Theorem 3.2. *Given a left state $v_L = (u_L, \sigma_L)$, the DLM path ϕ defined in* (1.3) *gives a shock-wave solution for the system* (1.2) *with the right states $v_R = (u_R, \sigma_R)$ on the shock curves*

$$S_1 : \sigma = \sigma_L - \frac{1}{8}((u - u_L)^2 - \sqrt{(u - u_L)^4 + 64k^2(u - u_L)^2}),\ u < u_L, \tag{3.2}$$

$$S_2 : \sigma = \sigma_L - \frac{1}{8}((u - u_L)^2 + \sqrt{(u - u_L)^4 + 64k^2(u - u_L)^2}),\ u < u_L, \tag{3.3}$$

Proof. Substituting ϕ_1 and ϕ_2 from (1.3) in (2.5) we obtain the following quadratic equation in $[\sigma]$:

$$4[\sigma]^2 + (u_R - u_L)^2[\sigma] - 4k^2(u_R - u_L)^2 = 0.$$

The above equation when solved gives us the required expressions for S_1 and S_2. □

Remark 3.3. *As k tends to* 0*, the shock curve S_2 defined in* (3.3) *for the system* (1.2) *tends to the shock curve for the system* (1.1) *given by* (3.1)*, while the other shock curve S_1 degenerates.*

Next we state the analogous results for the DLM path $\tilde{\phi}$ defined in (1.4).

Theorem 3.4. *Given a left state $v_L = (u_L, \sigma_L)$, the DLM path $\tilde{\phi}$ defined in* (1.4) *gives a shock-wave solution for the system* (1.1) *with the right states $v_R = (u_R, \sigma_R)$ on the shock curve*

$$S_1 : \sigma = \sigma_L - \frac{1}{2}(u - u_L)^2,\ u < u_L. \tag{3.4}$$

Proof. The result follows from a straightforward calculation by substituting $\tilde{\phi}_1$ and $\tilde{\phi}_2$ from (1.4) in (2.2). □

Theorem 3.5. *Given a left state $v_L = (u_L, \sigma_L)$, the DLM path $\tilde{\phi}$ defined in* (1.4) *gives a shock-wave solution for the system* (1.2) *with the right states $v_R = (u_R, \sigma_R)$ on the shock curves*

$$S_1 : \sigma = \sigma_L - \frac{1}{4}((u-u_L)^2 - \sqrt{(u-u_L)^4 + 16k^2(u-u_L)^2}),\ u < u_L, \tag{3.5}$$

$$S_2 : \sigma = \sigma_L - \frac{1}{4}((u-u_L)^2 + \sqrt{(u-u_L)^4 + 16k^2(u-u_L)^2}),\ u < u_L. \tag{3.6}$$

Proof. Substituting $\tilde{\phi}_1$ and $\tilde{\phi}_2$ from (1.4) in (2.5) we obtain the following quadratic equation in $[\sigma]$:

$$2[\sigma]^2 + (u_R - u_L)^2[\sigma] - 2k^2(u_R - u_L)^2 = 0.$$

The above equation when solved gives us the required expressions for S_1 and S_2. □

Remark 3.6. *As k tends to* 0*, the shock curve S_2 defined in* (3.6) *for the system* (1.2) *tends to the shock curve for the system* (1.1) *given by* (3.4)*, while the other shock curve S_1 degenerates.*

4. Conclusion

Thus we have shown the existence of shock-wave solutions for the systems (1.1) and (1.2) using DLM paths. This result is of more significance in the case of (1.1) due to the parabolic degeneracy it exhibits. We also remark that a rarefaction wave solution for the system (1.1) with Riemann type initial data is not possible. Therefore to solve the Riemann problem for arbitrary initial data we might have to consider the class of singular solutions. We refer to [3] for a discussion on singular solutions for the system (1.1). We have also shown the existence of more than one DLM paths which give rise to shock-wave solutions. As reflected here, different choices of DLM paths in general give rise to different solutions.

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